# A New Framework for Bounding Reachability Probabilities of Continuous-time Stochastic Systems

Bai Xue[1]



***Abstract*—This manuscript presents an innovative framework for constructing barrier functions to bound reachability probabilities for continuous-time stochastic systems described by stochastic differential equations (SDEs). The reachability probabilities considered in this paper encompass two aspects: the probability of reaching a set of specified states within a predefined finite time horizon, and the probability of reaching a set of specified states at a particular time instant. The barrier functions presented in this manuscript are developed either by relaxing a parabolic partial differential equation that characterizes the exact reachability probability or by applying the Grönwall's inequality. In comparison to the prevailing construction method, which relies on Doob's non-negative supermartingale inequality (or Ville's inequality), the proposed barrier functions provide stronger alternatives, complement existing methods, or fill gaps.**

## I. Introduction

Stochastic phenomena are commonly observed in both natural and artificial systems, spanning multiple disciplines such as biology and robotics. To accurately model these systems, sophisticated approaches are needed due to their inherent randomness [5]. Stochastic differential equations (SDEs) provide a powerful tool by integrating deterministic dynamics with stochastic processes, offering a comprehensive framework for comprehending the behavior of these systems [7]. They have been widely applied, such as in models of disturbances in engineered systems like wind forces [18] and pedestrian motion [6].

The reachability probability is a critical quantitative measure within the context of SDEs [10]. It provides valuable insights into the likelihood of a system, governed by an SDE, reaching a set of specified states within a predetermined (in)finite time frame (referred to as reachability probability within (in)finite time horizons) or at a particular time instant (referred to as reachability probability at specific time instants). This concept plays a pivotal role in understanding the probabilistic evolution of systems under stochastic influences, enabling informed analysis and decision-making in various fields. Computing reachability probabilities typically involves solving Hamilton-Jacobi-Bellman equations [8], [1], [3]. However, obtaining analytical solutions is often infeasible, necessitating the use of numerical approximations. As a result, obtaining both upper and lower bounds of reachability probabilities becomes impractical. [11] gave comparison results for SDEs that via a Lyapunov-like function allow reachability probabilities within finite time horizons to be upper-bounded by an exit probability of a one-dimensional Ornstein-Uhlenbeck process, but the bounds are not in closed form. Inspired by Lyapunov functions for stability analysis, determining upper and lower bounds of reachability probabilities within infinite and finite time horizons has been simplified by finding barrier certificates. With the development of polynomial optimization, specifically sum-of-squares polynomial optimization [12], barrier certificates have emerged as a powerful tool for certifying upper bounds of reachability probabilities. When the system of interest is polynomial, the problem of finding barrier certificates can be addressed through convex optimization. Barrier certificates for SDEs were initially introduced in [13], [14] for infinite time safety verification, upper-bounding the probability that a system will eventually reach an unsafe region based on a non-negative barrier function. They build upon the known Doob's non-negative supermartingale inequality [2], which requires the expectation of the non-negative barrier certificate to decrease along the system dynamics. Later, inspired by results in [9] and the Doob's non-negative supermartingale inequality, [16] extended barrier certificates to safety verification over finite time horizons and proposed c-martingales, which allow the expected value of the barrier function to increase over time. This approach provides upper bounds for the reachability probability of a system entering an unsafe region within finite time horizons. The c-martingales were further enhanced in [15] for safety verification over finite time horizons by imposing a state-dependent bound on the expected value of the barrier certificate. Recently, a controlled version was presented in [17]. Meanwhile, [4] proposed a time-varying barrier function to upper bound the reachability probability within finite time horizons, utilizing Doob's non-negative supermartingale inequality. While there has been considerable research focusing on providing upper bounds for the reachability probability within (in)finite time horizons, the practice of lower-bounding this probability has received considerably less attention. A novel equation, which can characterize the exact reachability probability within the infinite time horizon, was proposed in [22], [23]. By relaxing this equation, barrier-like conditions can be obtained to both lower-bound and upper-bound the reachability probability within the infinite time horizon. Recently, this approach has been extended to lower-bound the reachability probability within finite time horizons in [19], [20]. All of the aforementioned works study the bounding problem of reachability probabilities within (in)finite time horizons. However, to the best of our knowledge, there is no work in the framework of barrier functions investigating the problem of bounding reachability

1. State Key Lab. of Computer Science, Institute of Software Chinese Academy of Sciences, Beijing, China (xuebai@ios.ac.cn)

probabilities at specific time instants.

This paper explores the issue of lower- and upper-bounding reachability probabilities within finite time horizons and at specific time instants in stochastic systems modeled by SDEs. To tackle these problems, we propose time-dependent and time-independent barrier functions that provide lower and upper bounds for these reachability probabilities. The development of these barrier certificates is influenced by our previous work [23], [20], which introduces an alternative method that does not rely on the commonly used Doob's non-negative supermartingale inequality. Leveraging the occupation measure, the construction of the barrier certificates is achieved through either relaxation of a second-order partial differential equation or utilization of the Grönwall inequality. These barrier certificates are either more powerful compared to those found in previous works, complement the existing ones, or fill a gap. They fill facilitate the gain of tight bounds on reachability probabilities within finite time horizons and at specific time instants.

This paper is structured as follows. In Section II, we introduce SDEs and the problems related to bounding reachability probabilities within finite time horizons and at specific time instants. In Section III, we present our time-dependent and time-independent barrier functions for lower- and upper-bounding reachability probabilities within finite time horizons. Then, in Section IV, we present our barrier functions for lower- and upper-bounding reachability probabilities at specific time instants. Finally, in Section V, we conclude the paper.

Some basic notions are used throughout this paper: $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ stand for the set of real numbers and non-negative real numbers, respectively; $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the space of all $n$-dimensional vectors and $n\times m$ real matrices, respectively; for a set $\mathcal{A}$, the sets $\mathcal{A}^\circ$, $\overline{\mathcal{A}}$, and $\partial\mathcal{A}$ denote the interior, the closure, and the boundary of the set $\mathcal{A}$, respectively; $\wedge$ denotes the logical operation of conjunction.

## II. Preliminaries

This section introduces SDEs and the reachability probabilities bounding problem of interest.

Consider the continuous-time stochastic system,

$$d\boldsymbol{x}(t,\boldsymbol{w}) = \boldsymbol{b}(\boldsymbol{x}(t,\boldsymbol{w}))dt + \boldsymbol{\sigma}(\boldsymbol{x}(t,\boldsymbol{w}))d\boldsymbol{W}(t,\boldsymbol{w}), \tag{1}$$

where $\boldsymbol{b}(\cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{\sigma}(\cdot)\colon \mathbb{R}^n \to \mathbb{R}^{n\times k}$ are locally Lipschitz continuous function; $\boldsymbol{W}(t,\boldsymbol{w})\colon \mathbb{R}\times\Omega \to \mathbb{R}^k$ is an $k$-dimensional Wiener process (standard Brownian motion), and $\Omega$, equipped with the probability measure $\mathbb{P}$, is the sample space $\boldsymbol{w}$ belongs to. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}[\cdot]$.

Given an initial state $\boldsymbol{x}_0$, the SDE (1) has a unique (maximal local) strong solution over a time interval $[0, T^{\boldsymbol{x}_0}(\boldsymbol{w}))$, where $T^{\boldsymbol{x}_0}(\boldsymbol{w})$ is a positive real value or infinity. This solution is denoted as $\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\cdot)\colon [0, T^{\boldsymbol{x}_0}(\boldsymbol{w})) \to \mathbb{R}^n$, which satisfies the stochastic integral equation,

$$\begin{aligned}\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t) = \boldsymbol{x}_0 &+ \int_0^t \boldsymbol{b}(\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\tau))d\tau \\ &+ \int_0^t \boldsymbol{\sigma}(\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\tau))d\boldsymbol{W}(\tau,\boldsymbol{w}).\end{aligned}$$

Given a function $v(t,\boldsymbol{x})$ that is twice continuously differentiable over $\boldsymbol{x}$ and continuously differentiable over $t$, the infinitesimal generator underlying system (1), which represents the limit of the expected value of $v(t, \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t))$ as $t$ approaches 0, is presented in Definition 1.

**Definition 1.** *Given system* (1)*, the infinitesimal generator of a function $v(t,\boldsymbol{x})$ that is twice continuously differentiable over $\boldsymbol{x}$ and continuously differentiable over $t$ is defined by*

$$\begin{aligned}\mathcal{L}v(t,\boldsymbol{x}) &= \lim_{\Delta t\to 0}\frac{\mathbb{E}[v(t+\Delta t, \boldsymbol{X}_{\boldsymbol{x}}^{\boldsymbol{w}}(t+\Delta t))] - v(t,\boldsymbol{x})}{t}\\ &= \frac{\partial v}{\partial t} + \frac{\partial v}{\partial \boldsymbol{x}}\boldsymbol{b}(\boldsymbol{x}) + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}(\boldsymbol{x})^\top \frac{\partial^2 v}{\partial \boldsymbol{x}^2}\boldsymbol{\sigma}(\boldsymbol{x})),\end{aligned}$$

*where $\frac{\partial v}{\partial t}$ and $\frac{\partial v}{\partial \boldsymbol{x}}$ represent the gradient of the function $v(t,\boldsymbol{x})$ with respect to $t$ and $\boldsymbol{x}$, respectively, $\frac{\partial^2 v}{\partial \boldsymbol{x}^2}$ represents the second-order partial derivative of the function $v(t,\boldsymbol{x})$ with respect to $\boldsymbol{x}$, and* $\mathrm{tr}(\cdot)$ *denotes the trace of a matrix.*

Given a state constrained set $\mathcal{X} \subseteq \mathbb{R}^n$ that is open and bounded, and a subset $\mathcal{X}_s \subseteq \mathcal{X}$ that is closed, the reachability probability within a time horizon $[0,T]$ with $0 < T < \infty$ is the probability of system (1), starting from an initial state $\boldsymbol{x}_0 \in \mathcal{X}\setminus\mathcal{X}_s$, reaches the set $\mathcal{X}_s$ within the time horizon $[0,T]$ while remaining within the state constrained set $\mathcal{X}$ until the first occurrence of hitting the set $\mathcal{X}_s$. It is formulated in Definition 2.

**Definition 2** (Reachability Probability I)**.** *Given a time horizon $[0,T]$ with $0<T<\infty$ and an initial state $\boldsymbol{x}_0 \in \mathcal{X}\setminus\mathcal{X}_s$, the reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}$ within the time horizon $[0,T]$ for system* (1) *starting from an initial state $\boldsymbol{x}_0 \in \mathcal{X}\setminus\mathcal{X}_s$ is defined below:*

$$\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]} := \mathbb{P}\left(\left\{\boldsymbol{w}\in\Omega \,\middle|\, \begin{array}{l}\exists t\in[0,T].\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t)\in\mathcal{X}_s\\ \wedge\,\forall\tau\in[0,t).\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\tau)\in\mathcal{X}\end{array}\right\}\right).$$

Given a state constrained set $\mathcal{X} \subseteq \mathbb{R}^n$ that is open and bounded, and a subset $\mathcal{X}_s \subseteq \mathcal{X}$ that is closed, the reachability probability at a time instant $T$ with $0 < T < \infty$ is the probability of system (1), starting from an initial state $\boldsymbol{x}_0 \in \mathcal{X}\setminus\mathcal{X}_s$, reaches the set $\mathcal{X}_s$ at the time instant $T$ while remaining within the state constrained set $\mathcal{X}$ before the time $T$. It is formulated in Definition 3.

**Definition 3** (Reachability Probability II)**.** *Given a time horizon $[0,T]$ with $0<T<\infty$ and an initial state $\boldsymbol{x}_0 \in \mathcal{X}\setminus\mathcal{X}_s$, the reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^{T}$ at the time instant $T>0$ for system* (1) *starting from an initial state $\boldsymbol{x}_0 \in \mathcal{X}\setminus\mathcal{X}_s$ is defined below:*

$$\mathbb{P}_{\boldsymbol{x}_0}^{T} := \mathbb{P}\left(\left\{\boldsymbol{w}\in\Omega \,\middle|\, \begin{array}{l}\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T)\in\mathcal{X}_s\wedge\\ \forall\tau\in[0,T).\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\tau)\in\mathcal{X}\end{array}\right\}\right).$$

In this paper, we address the challenge of computing the exact reachability probabilities in Definition 2 and 3, which is often infeasible for nonlinear systems. Instead, we resort to characterizing their lower and upper bounds, i.e., we will characterize $\delta_{i,1}\in[0,1]$ and $\delta_{i,2}\in[0,1]$, $i=1,2$, such that

$$\delta_{1,1} \leq \mathbb{P}_{\boldsymbol{x}_0}^{[0,T]} \leq \delta_{1,2}$$

and

$$\delta_{2,1} \le \mathbb{P}_{\boldsymbol{x}_0}^T \le \delta_{2,2}.$$

## III. Bounding Reachability Probabilities I

This section introduces our barrier functions for upper- and lower-bounding the reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}$ in Definition 2. They are respectively formulated in Subsection III-A and III-B.

The construction of these barrier functions lies on an auxiliary stochastic process $\{\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t), t \in \mathbb{R}_{\ge 0}\}$ for $\boldsymbol{x}_0 \in \overline{\mathcal{X} \setminus \mathcal{X}_s}$ that is a stopped process corresponding to $\{\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t), t \in [0, T^{\boldsymbol{x}_0}(\boldsymbol{w}))\}$ and the set $\overline{\mathcal{X} \setminus \mathcal{X}_s}$, i.e.,

$$\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t) = \begin{cases} \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t) & \text{if } t < \tau^{\boldsymbol{x}_0}(\boldsymbol{w}) \\ \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\tau^{\boldsymbol{x}_0}(\boldsymbol{w})) & \text{if } t \ge \tau^{\boldsymbol{x}_0}(\boldsymbol{w}) \end{cases}, \tag{2}$$

where $\tau^{\boldsymbol{x}_0}(\boldsymbol{w}) = \inf\{\, t \mid \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t) \in \partial\mathcal{X} \cup \partial\mathcal{X}_s \,\}$ is the first time of exit of $\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t)$ from the set $\mathcal{X} \setminus \mathcal{X}_s$. It is worth remarking here that if the path $\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t)$ escapes to infinity in finite time, it must touch $\partial\mathcal{X} \cup \partial\mathcal{X}_s$ and thus $\tau^{\boldsymbol{x}_0}(\boldsymbol{w}) \le T^{\boldsymbol{x}_0}(\boldsymbol{w})$. The stopped process $\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t)$ inherits the right continuity and strong Markovian property of $\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t)$. Moreover, the infinitesimal generator corresponding to $\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t)$ is identical to the one corresponding to $\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t)$ over $\mathcal{X} \setminus \mathcal{X}_s$, and is equal to $\frac{\partial v(t,\boldsymbol{x})}{\partial t}$ on $\partial\mathcal{X} \cup \partial\mathcal{X}_s$ [9]. That is, for $v(t, \boldsymbol{x})$ that is twice continuously differentiable over $\boldsymbol{x}$ and continuously differentiable over $t$,

$$\begin{aligned} \widehat{\mathcal{L}}v(t,\boldsymbol{x}) = \mathcal{L}v(t,\boldsymbol{x}) = & \frac{\partial v(t,\boldsymbol{x})}{\partial t} + \frac{\partial v(t,\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{b}(\boldsymbol{x}) \\ & + \frac{1}{2} \text{tr}(\boldsymbol{\sigma}(\boldsymbol{x})^\top \frac{\partial^2 v(t,\boldsymbol{x})}{\partial \boldsymbol{x}^2} \boldsymbol{\sigma}(\boldsymbol{x})) \end{aligned}$$

for $(\boldsymbol{x}, t) \in \mathcal{X} \setminus \mathcal{X}_s \times [0, T]$ and $\widehat{\mathcal{L}}v(t, \boldsymbol{x}) = \frac{\partial v(t,\boldsymbol{x})}{\partial t}$ for $\boldsymbol{x} \in \partial\mathcal{X} \cup \partial\mathcal{X}_s$ and $t \in [0, T]$.

Given an initial state $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$, the exact reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}$ is equal to the probability of reaching the set $\partial\mathcal{X}_s$ at the time instant $T$ for the above auxiliary stochastic process. Before justifying this statement, we first show that the reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}$ is equal to the probability that system (1) will reach the boundary $\partial\mathcal{X}_s$ of the set $\mathcal{X}_s$ within the time horizon $[0, T]$ while remaining within the state constrained set $\mathcal{X}$ until the first occurrence of hitting the set $\partial\mathcal{X}_s$.

**Lemma 1.** *Given* $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$,

$$\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]} = \mathbb{P}\left( \left\{ \boldsymbol{w} \in \Omega \,\middle|\, \begin{array}{l} \exists t \in [0,T]. \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t) \in \partial\mathcal{X}_s \wedge \\ \forall \tau \in [0,t). \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\tau) \in \mathcal{X} \setminus \mathcal{X}_s \end{array} \right\} \right).$$

*Proof.* Since $\mathcal{X}_s$ is a closed set, we have $A = B$, where $A = \{\, \boldsymbol{w} \in \Omega \mid \exists t \in [0, T]. \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t) \in \partial\mathcal{X}_s \bigwedge \forall \tau \in [0, t). \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\tau) \in \mathcal{X} \,\}$ and $B = \{\, \boldsymbol{w} \in \Omega \mid \exists t \in [0, T]. \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t) \in \mathcal{X}_s \bigwedge \forall \tau \in [0, t). \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\tau) \in \mathcal{X} \,\}$. Thus, we have the conclusion. ■ □

**Lemma 2** (Lemma 1, [20])**.** *Given* $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$,

$$\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]} = \mathbb{P}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T) \in \partial\mathcal{X}_s) = \mathbb{E}[1_{\partial\mathcal{X}_s}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))].$$

*Moreover, for any* $0 < T_1 \le T_2 \le T$,

$$\mathbb{P}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T_1) \in \partial\mathcal{X}_s) \le \mathbb{P}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T_2) \in \partial\mathcal{X}_s).$$

Further, the exact reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}$ can be reduced to a solution to a second-order partial differential equation.

**Theorem 1.** *Suppose there exists a function* $v(t, \boldsymbol{x})\colon [0, T] \times \mathbb{R}^n$ *which is twice continuously differentiable over* $\boldsymbol{x}$ *and continuously differentiable over* $t$*, satisfying*

$$\begin{cases} \widehat{\mathcal{L}}v(t,\boldsymbol{x}) = 0 & \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_s}, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) = 1_{\partial\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_s} \end{cases}, \tag{3}$$

*then,* $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]} = v(0, \boldsymbol{x}_0)$ *for* $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$.

*Proof.* The proof relies on Dynkin's formula and Lemma 2: for $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$, we obtain

$$\begin{aligned} \mathbb{P}_{\boldsymbol{x}_0}^{[0,T]} &= \mathbb{P}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T) \in \partial\mathcal{X}_s) \\ &= \mathbb{E}[1_{\partial\mathcal{X}_s}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))] \\ &= \mathbb{E}[v(T, \widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))] \\ &= v(0,\boldsymbol{x}_0) + \mathbb{E}[\int_0^T \widehat{\mathcal{L}}v(t, \widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t)) dt] \\ &= v(0,\boldsymbol{x}_0). \end{aligned}$$

The proof is completed. ■ □

### A. Upper-bounding Reachability Probabilities

In this subsection, we present our barrier functions for upper-bounding the reachability probability, denoted as $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}$ in Definition 2. The first time-dependent barrier function is obtained via relaxing equation (3) as stated in Theorem 1. The second one extends upon the first one, which relaxes the supermartingale requirement. The third one is a variant of the second one, using a time-independent function $v(\boldsymbol{x})$ instead of a time-dependent function $v(t, \boldsymbol{x})$. They are respectively formulated in Lemma 3, 4, and Corollary 1.

**Lemma 3.** *Suppose there exists a barrier function* $v(t, \boldsymbol{x})\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *that is continuously differentiable over* $t$ *and twice continuously differentiable over* $\boldsymbol{x}$*, satisfying*

$$\begin{cases} \mathcal{L}v(t,\boldsymbol{x}) \le 0, & \forall \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_s, \forall t \in [0,T] \\ \frac{\partial v(t,\boldsymbol{x})}{\partial t} \le 0 & \forall \boldsymbol{x} \in \partial\mathcal{X} \cup \partial\mathcal{X}_s, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) \ge 1_{\partial\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_s}, \end{cases}, \tag{4}$$

*then,* $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]} \le v(0, \boldsymbol{x}_0)$ *for* $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$.

*Proof.* From (4), we have

$$\begin{cases} \widehat{\mathcal{L}}v(t,\boldsymbol{x}) \le 0 & \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_s}, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) \ge 1_{\partial\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X} \setminus \mathcal{X}_s} \end{cases}. \tag{5}$$

Based on (5), we can obtain the conclusion using the Dynkin's formula and Lemma 2: for $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$, we obtain

$$\begin{aligned} \mathbb{P}_{\boldsymbol{x}_0}^{[0,T]} &= \mathbb{P}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T) \in \partial\mathcal{X}_s) \\ &= \mathbb{E}[1_{\partial\mathcal{X}_s}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))] \\ &\le \mathbb{E}[v(T, \widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))] \\ &= v(0,\boldsymbol{x}_0) + \mathbb{E}[\int_0^T \widehat{\mathcal{L}}v(t, \widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t)) dt] \\ &\le v(0,\boldsymbol{x}_0). \end{aligned}$$

The proof is completed. ■ □

We found another time-dependent barrier function for upper-bounding the reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}$ in Theorem 5 in [4], which is formulated below: Suppose there exists a constant $\eta > 0$ and a barrier function $v(t,\boldsymbol{x})\colon \mathbb{R}\times\mathbb{R}^n$, satisfying

$$\begin{cases}\mathcal{L}v(t,\boldsymbol{x})\le 0 & \forall \boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s, \forall t\in[0,T]\\ \frac{\partial v(t,\boldsymbol{x})}{\partial t}\le 0 & \forall \boldsymbol{x}\in\partial\mathcal{X}, \forall t\in[0,T]\\ v(t,\boldsymbol{x})\ge \eta 1_{\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x}\in\overline{\mathcal{X}}, \forall t\in[0,T]\end{cases}, \quad (6)$$

then, $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}\le\frac{v(0,\boldsymbol{x}_0)}{\eta}$.

Upon comparing conditions (4) and (6), it becomes apparent that condition (6) imposes the requirement of non-negativity for the barrier function $v(t,\boldsymbol{x})$ over $[0,T]\times\overline{\mathcal{X}}$. Conversely, condition (4) solely necessitates non-negativity for the function $v(t,\boldsymbol{x})$ over $\overline{\mathcal{X}\setminus\mathcal{X}_s}$ at $t=T$. These disparities arise due to the construction of (6) using the well-established Doob's non-negative supermartingale inequality (also known as Ville's inequality [2]). In contrast, condition (4) is formulated by relaxing equation (3).

Lemma 3 states that if there exists a barrier function $v(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n\to\mathbb{R}$ satisfying (4), then the reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}$ can be bounded above by $v(0,\boldsymbol{x}_0)$. However, the requirement for $\mathcal{L}v(t,\boldsymbol{x})\le 0$ to hold over $(t,\boldsymbol{x})\in[0,T]\times(\mathcal{X}\setminus\mathcal{X}_s)$ may hinder the acquisition of such a barrier function. In the following, we will further relax this supermartingale requirement.

**Lemma 4.** *Suppose there exists a barrier function $v(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n\to\mathbb{R}$ that is continuously differentiable over $t$ and twice continuously differentiable over $\boldsymbol{x}$, satisfying*

$$\begin{cases}\mathcal{L}v(t,\boldsymbol{x})\le \alpha v(t,\boldsymbol{x})+\beta & \forall \boldsymbol{x}\in\mathcal{X}\setminus\ \mathcal{X}_s, \forall t\in[0,T]\\ \frac{\partial v(t,\boldsymbol{x})}{\partial t}\le \alpha v(t,\boldsymbol{x})+\beta & \forall \boldsymbol{x}\in\partial\mathcal{X}\cup\partial\mathcal{X}_s, \forall t\in[0,T]\\ v(T,\boldsymbol{x})\ge 1_{\partial\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_s}\end{cases}. \quad (7)$$

*then,*

$$\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}\le\begin{cases}v(0,\boldsymbol{x}_0)+\beta T & \text{if } \alpha=0\\ e^{\alpha T}v(0,\boldsymbol{x}_0)+\frac{\beta}{\alpha}(e^{\alpha T}-1) & \text{if } \alpha\neq 0\end{cases}$$

*for $\boldsymbol{x}_0\in\mathcal{X}\setminus\mathcal{X}_s$.*

*Proof.* From (7), we have

$$\begin{cases}\widehat{\mathcal{L}}v(t,\boldsymbol{x})\le \alpha v(t,\boldsymbol{x})+\beta & \forall \boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_s}, \forall t\in[0,T]\\ v(T,\boldsymbol{x})\ge 1_{\partial\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_s}\end{cases}.$$

If $\alpha=0$, we have that

$$\begin{aligned}\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}&=\mathbb{P}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T)\in\partial\mathcal{X}_s)\\ &=\mathbb{E}[1_{\partial\mathcal{X}_s}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))]\\ &\le\mathbb{E}[v(T,\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))]\\ &=v(0,\boldsymbol{x}_0)+\mathbb{E}[\int_0^T\mathcal{L}v(t,\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t))dt]\\ &\le v(0,\boldsymbol{x}_0)+\beta T.\end{aligned}$$

When $\alpha\neq 0$, we have, by the Grönwall's inequality in the differential form, that

$$\mathbb{E}[v(T,\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))]\le e^{\alpha T}v(0,\boldsymbol{x}_0)+\frac{\beta}{\alpha}(e^{\alpha T}-1).$$

Also, since $v(T,\boldsymbol{x})\ge 1_{\partial\mathcal{X}_s}(\boldsymbol{x}), \forall\boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_s}$, we have $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}=\mathbb{E}[1_{\partial\mathcal{X}_s}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))]\le\mathbb{E}[v(T,\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))]$ and thus $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}=\mathbb{P}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T)\in\partial\mathcal{X}_s)\le e^{\alpha T}v(0,\boldsymbol{x}_0)+\frac{\beta}{\alpha}(e^{\alpha T}-1)$.

The proof is completed. ■ □

It is easy to observe that condition (4) in Lemma 3 is a special case of the one (7) in Lemma 4 with $\alpha=\beta=0$. Below, a straightforward variant of the result in Lemma 4 is presented. Instead of resorting to a time-dependent barrier function $v(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n\to\mathbb{R}$, the focus is on finding a time-independent one $v(\boldsymbol{x})\colon\mathbb{R}^n\to\mathbb{R}$.

**Corollary 1.** *Suppose there exists a barrier function $v(\boldsymbol{x})\colon\mathbb{R}^n\to\mathbb{R}$, which is twice continuously differentiable, satisfying*

$$\begin{cases}\mathcal{L}v(\boldsymbol{x})\le\alpha v(\boldsymbol{x})+\beta & \forall\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s\\ 0\le\alpha v(\boldsymbol{x})+\beta & \forall\boldsymbol{x}\in\partial\mathcal{X}\cup\partial\mathcal{X}_s\\ v(\boldsymbol{x})\ge 1_{\partial\mathcal{X}_s}(\boldsymbol{x}) & \forall\boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_s}\end{cases}, \quad (8)$$

*then,*

$$\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}\le\begin{cases}v(\boldsymbol{x}_0)+\beta T & \text{if } \alpha=0\\ e^{\alpha T}v(\boldsymbol{x}_0)+\frac{\beta}{\alpha}(e^{\alpha T}-1) & \text{if } \alpha\neq 0\end{cases}$$

*for $\boldsymbol{x}_0\in\mathcal{X}\setminus\mathcal{X}_s$.*

From $v(\boldsymbol{x})\ge 1_{\partial\mathcal{X}_s}(\boldsymbol{x}), \forall\boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_s}$ in (8), we can obtain $v(\boldsymbol{x})\ge 0$ for $\boldsymbol{x}\in\partial\mathcal{X}$ and $v(\boldsymbol{x})\ge 1$ for $\boldsymbol{x}\in\partial\mathcal{X}_s$. Since $0\le\alpha v(\boldsymbol{x})+\beta, \forall\boldsymbol{x}\in\partial\mathcal{X}\cup\partial\mathcal{X}_s$, we can obtain $\alpha+\beta\ge 0$ if $\alpha\le 0$. However, it is worth remarking here that $\alpha>0$ is permitted in condition (8).

Another upper bound of $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}$ was derived in [15]: if $v(\boldsymbol{x})$ satisfies

$$\begin{cases}\mathcal{L}v(\boldsymbol{x})\le\alpha v(\boldsymbol{x})+\beta & \forall\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s\\ v(\boldsymbol{x})\ge 1 & \forall\boldsymbol{x}\in\mathcal{X}_s\\ v(\boldsymbol{x})\ge 0 & \forall\boldsymbol{x}\in\mathcal{X}\end{cases}, \quad (9)$$

an upper bound of $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}$ is

$$\begin{cases}(v(\boldsymbol{x}_0)-(e^{\beta T}-1)\frac{\beta}{\alpha})e^{-\beta T} & \text{if } \alpha<0\wedge\alpha+\beta>0\\ v(\boldsymbol{x}_0)+\beta T & \text{if } \alpha=0\wedge\beta\ge 0\\ e^{-\beta T}(v(\boldsymbol{x}_0)-1)+1 & \text{if } \alpha<0\wedge\alpha+\beta\le 0\wedge\beta\ge 0\end{cases}.$$

These results are obtained via following Theorem 1 in Chapter 3 in [9] and are built upon the known Doob's nonnegative supermartingale inequality (or, Ville's inequality [2]) as condition (6). However, it is observed that as $\alpha$ approaches $0^-$, $(v(\boldsymbol{x}_0)-(e^{\beta T}-1)\frac{\beta}{\alpha})e^{-\beta T}$ is not equal to $v(\boldsymbol{x}_0)+\beta T$ as expected. In contrast, it tends to infinity, which is overly conservative. Condition (9) in Corollary 1 has certain advantages over (9): firstly, as $\alpha$ approaches $0^-$, the expression $e^{\alpha T}v(\boldsymbol{x}_0)+\frac{\beta}{\alpha}(e^{\alpha T}-1)$ converges to $v(\boldsymbol{x}_0)+\beta T$; secondly, when $v(\boldsymbol{x}_0)\le 1<-\frac{\beta}{\alpha}$ (when $\alpha<0$, $1<-\frac{\beta}{\alpha}$ implies $\alpha+\beta>0$), we can obtain $(v(\boldsymbol{x}_0)-(e^{\beta T}-1)\frac{\beta}{\alpha})e^{-\beta T}>$

$e^{\alpha T}v(\boldsymbol{x}_0)+\frac{\beta}{\alpha}(e^{\alpha T}-1)$; thirdly, unlike the aforementioned condition (9), condition (8) is not restricted to the scenario where $\alpha\leq 0$ and/or $\beta\geq 0$.

### B. Lower-bounding Reachability Probabilities

In this subsection, we present our barrier functions for lower-bounding the reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}$, based on the following assumption on $\mathcal{X}_s$.

**Assumption 1.** *The set $\mathcal{X}_s$ has non-empty interior, i.e., $\mathcal{X}_s^{\circ}\neq\emptyset$.*

The construction of the first barrier function was inspired by [22], [23]. It cannot be obtained via relaxing equation (3) directly. An auxiliary function is introduced.

**Lemma 5.** *Suppose there exist a barrier function $v(t,\boldsymbol{x})\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}$ and a function $w(t,\boldsymbol{x})\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}$ with $\sup_{(t,\boldsymbol{x})\in[0,T]\times\overline{\mathcal{X}}}|w(t,\boldsymbol{x})|\leq M$ that are continuously differentiable over $t$ and twice continuously differentiable over $\boldsymbol{x}$, satisfying*

$$\begin{cases}\mathcal{L}v(t,\boldsymbol{x})\geq 0 & \forall \boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s,\forall t\in[0,T]\\ \frac{\partial v(t,\boldsymbol{x})}{\partial t}\geq 0 & \forall \boldsymbol{x}\in\partial\mathcal{X}\cup\partial\mathcal{X}_s,\forall t\in[0,T]\\ v(t,\boldsymbol{x})\leq 1+\frac{\partial w(t,\boldsymbol{x})}{\partial t} & \forall \boldsymbol{x}\in\partial\mathcal{X}_s,\forall t\in[0,T]\\ v(t,\boldsymbol{x})\leq \mathcal{L}w(t,\boldsymbol{x}) & \forall \boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s,\forall t\in[0,T]\\ v(t,\boldsymbol{x})\leq \frac{\partial w(t,\boldsymbol{x})}{\partial t} & \forall \boldsymbol{x}\in\partial\mathcal{X},\forall t\in[0,T]\end{cases}, \tag{10}$$

*then, $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}\geq v(0,\boldsymbol{x}_0)-\frac{2M}{T}$ for $\boldsymbol{x}_0\in\mathcal{X}\setminus\mathcal{X}_s$.*

*Proof.* From (10), we have

$$\begin{cases}\widehat{\mathcal{L}}v(t,\boldsymbol{x})\geq 0 & \forall\boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_s},\forall t\in[0,T]\\ v(t,\boldsymbol{x})\leq 1_{\partial\mathcal{X}_s}(\boldsymbol{x})+\widehat{\mathcal{L}}w(t,\boldsymbol{x}) & \forall\boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_s},\forall t\in[0,T]\end{cases}.$$

According to $\widehat{\mathcal{L}}v(t,\boldsymbol{x})\geq 0,\forall\boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_s},\forall t\in[0,T]$, we have, for $t\in[0,T]$, that

$$\begin{aligned}\mathbb{E}[v(t,\widehat{\boldsymbol{X}}^{\boldsymbol{w}}_{\boldsymbol{x}_0}(t))]&=v(0,\boldsymbol{x}_0)+\mathbb{E}[\int_0^t\widehat{\mathcal{L}}v(\tau,\widehat{\boldsymbol{X}}^{\boldsymbol{w}}_{\boldsymbol{x}_0}(t))d\tau]\\&\geq v(0,\boldsymbol{x}_0).\end{aligned}$$

Further, from $v(t,\boldsymbol{x})\leq 1_{\partial\mathcal{X}_s}(\boldsymbol{x})+\widehat{\mathcal{L}}w(t,\boldsymbol{x}),\forall\boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_s},\forall t\in[0,T]$, we have

$$\begin{aligned}&\mathbb{P}(\widehat{\boldsymbol{X}}^{\boldsymbol{w}}_{\boldsymbol{x}_0}(T)\in\partial\mathcal{X}_s)\\&=\mathbb{E}[1_{\partial\mathcal{X}_s}(\widehat{\boldsymbol{X}}^{\boldsymbol{w}}_{\boldsymbol{x}_0}(T))]\\&\geq\frac{\int_0^T\mathbb{E}[1_{\partial\mathcal{X}_s}(\widehat{\boldsymbol{X}}^{\boldsymbol{w}}_{\boldsymbol{x}_0}(t))dt]}{T}\text{(according to Lemma 2)}\\&\geq\frac{\int_0^T\mathbb{E}[v(t,\widehat{\boldsymbol{X}}^{\boldsymbol{w}}_{\boldsymbol{x}_0}(t))]dt}{T}-\frac{\mathbb{E}[w(T,\widehat{\boldsymbol{X}}^{\boldsymbol{w}}_{\boldsymbol{x}_0}(T))]-w(0,\boldsymbol{x}_0)}{T}\\&\geq v(0,\boldsymbol{x}_0)-\frac{\mathbb{E}[w(T,\widehat{\boldsymbol{X}}^{\boldsymbol{w}}_{\boldsymbol{x}_0}(T))]-w(0,\boldsymbol{x}_0)}{T}\\&\geq v(0,\boldsymbol{x}_0)-\frac{2M}{T}.\end{aligned}$$

The proof is completed. ■ □

**Remark 1.** *The function $w(t,\boldsymbol{x})$ in Lemma 5 cannot be removed. If this function is removed, we cannot find a barrier function $v(t,\boldsymbol{x})$ satisfying condition* (10) *such that the lower bound $v(0,\boldsymbol{x}_0)$ of the reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]}$ is larger than zero. A brief explanation is given here: if the function $w(t,\boldsymbol{x})$ is removed, $v(t,\boldsymbol{x})$ will satisfy*

$$\begin{cases}v(t,\boldsymbol{x})\leq 1 & \forall\boldsymbol{x}\in\partial\mathcal{X}_s,\forall t\in[0,T]\\ v(t,\boldsymbol{x})\leq 0 & \forall\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s,\forall t\in[0,T]\end{cases},$$

*which implies $v(t,\boldsymbol{x})\leq 0,\forall\boldsymbol{x}\in\overline{\mathcal{X}\setminus\mathcal{X}_s},\forall t\in[0,T]$, thus, $v(0,\boldsymbol{x}_0)$ will always be less than or equal to zero for $\boldsymbol{x}_0\in\mathcal{X}\setminus\mathcal{X}_s$.*

**Remark 2.** *The barrier function for certifying reach-avoid analysis over the time horizon $[0,T]$ (i.e., reach the set $\mathcal{X}_r$ at some time instant $t\in[0,T]$ while staying within the set $\mathcal{X}$ before $t$) for deterministic systems can be retrieved from the one satisfying* (10).

*When $\boldsymbol{\sigma}(\cdot)\equiv\boldsymbol{0}$, stochastic system* (1) *degenerates into a deterministic system*

$$\frac{d\boldsymbol{x}}{dt}=\boldsymbol{b}(\boldsymbol{x}), \tag{11}$$

*where $\boldsymbol{X}_{\boldsymbol{x}_0}(\cdot)\colon[0,T^{\boldsymbol{x}_0})\to\mathbb{R}^n$ is the solution with the initial state $\boldsymbol{x}_0$ at $t=0$.*

*If there exist a barrier function $v(t,\boldsymbol{x})\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}$ and a function $w(t,\boldsymbol{x})\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}$ with $\sup_{(t,\boldsymbol{x})\in[0,T]\times\overline{\mathcal{X}}}|w(t,\boldsymbol{x})|\leq M$ that are continuously differentiable over $t$ and $\boldsymbol{x}$, satisfying condition* (10)*, then the system described by equation* (11)*, when initiated from the set $\{\,\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s\mid v(0,\boldsymbol{x})>\frac{2M}{T}\,\}$, will reach the set $\mathcal{X}_s$ within the time interval $[0,T]$, while remaining within the set $\mathcal{X}$ before the first encounter with $\mathcal{X}_s$. A proof of this conclusion is shown here: we first demonstrate that if system* (11) *is initiated from the set $\{\,\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s\mid v(0,\boldsymbol{x})>\frac{2M}{T}\,\}$, within the time interval $[0,T]$, it remains within the set $\mathcal{X}$ if it does not reach the set $\mathcal{X}_s$. Assume it is not true. Then there exists $\tau\in[0,T]$ such that $\boldsymbol{X}_{\boldsymbol{x}_0}(\tau)\in\partial\mathcal{X}$ and $\boldsymbol{X}_{\boldsymbol{x}_0}(t)\in\mathcal{X}\setminus\mathcal{X}_s$ for $t\in[0,\tau)$. Since $\mathcal{L}v(t,\boldsymbol{x})\geq 0$ for $t\in[0,T]$ and $\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s$, we have*

$$v(\tau,\boldsymbol{X}_{\boldsymbol{x}_0}(\tau))\geq v(0,\boldsymbol{x}_0)>\frac{2M}{T}.$$

*Also, according to $v(t,\boldsymbol{x})\leq\mathcal{L}w(t,\boldsymbol{x})$ for $(t,\boldsymbol{x})\in[0,T]\times\mathcal{X}\setminus\mathcal{X}_s$, we have*

$$\frac{2M}{T}\tau<\int_0^\tau v(t,\boldsymbol{x})dt\leq w(\tau,\boldsymbol{X}_{\boldsymbol{x}_0}(\tau))-w(0,\boldsymbol{x}_0)$$

*and thus $w(\tau,\boldsymbol{X}_{\boldsymbol{x}_0}(\tau))>\frac{2M}{T}\tau+w(0,\boldsymbol{x}_0)$. Further, since $\frac{\partial v(t,\boldsymbol{x})}{\partial t}\geq 0$ and $v(t,\boldsymbol{x})\leq\frac{\partial w(t,\boldsymbol{x})}{\partial t}$ for $(t,\boldsymbol{x})\in[0,T]\times\partial\mathcal{X}$, we have $v(\tau,\boldsymbol{X}_{\boldsymbol{x}_0}(\tau))(T-\tau)\leq\int_\tau^T v(t,\boldsymbol{X}_{\boldsymbol{x}_0}(\tau))dt\leq\int_\tau^T\frac{\partial w(t,\boldsymbol{X}_{\boldsymbol{x}_0}(\tau))}{\partial t}dt=w(T,\boldsymbol{X}_{\boldsymbol{x}_0}(\tau))-w(\tau,\boldsymbol{X}_{\boldsymbol{x}_0}(\tau))$. Thus,*

$$\begin{aligned}\frac{2M}{T}(T-\tau)&<v(\tau,\boldsymbol{X}_{\boldsymbol{x}_0}(\tau))(T-\tau)\\&\leq w(T,\boldsymbol{X}_{\boldsymbol{x}_0}(\tau))-\frac{2M}{T}\tau-w(0,\boldsymbol{x}_0)\end{aligned} \tag{12}$$

*holds and consequently, $2M<w(T,\boldsymbol{X}_{\boldsymbol{x}_0}(\tau))-w(0,\boldsymbol{x}_0)\leq 2M$, which is a contradiction. Therefore, the system* (11)*, when initiated from the set $\{\,\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s\mid v(0,\boldsymbol{x})>\frac{2M}{T}\,\}$, will not reach the set $\partial\mathcal{X}$ within the time interval $[0,T]$ if it does not* *enter the set $\mathcal{X}_s$. Next, we show that the system* (11)*, when initiated from the set* $\{\, \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_s \mid v(0,\boldsymbol{x}) > \frac{2M}{T} \,\}$*, will reach the set $\mathcal{X}_s$ within the time interval $[0,T]$. Assume that this conclusion does not hold, which indicates that the system when initiated from the set* $\{\, \boldsymbol{x} \in \mathcal{X}\setminus\mathcal{X}_s \mid v(0,\boldsymbol{x}) > \frac{2M}{T} \,\}$*, will stay within the set $\mathcal{X} \setminus \mathcal{X}_s$ over the time horizon $[0,T]$. From $v(t,\boldsymbol{x}) \leq \mathcal{L}w(t,\boldsymbol{x})$ for $(t,\boldsymbol{x}) \in [0,T]\times\mathcal{X}\setminus\mathcal{X}_s$, we can obtain* $\frac{2M}{T}T < \int_0^T v(t,\boldsymbol{X}_{\boldsymbol{x}_0}(t))dt \leq w(T,\boldsymbol{X}_{\boldsymbol{x}_0}(T)) - w(0,\boldsymbol{x}_0) \leq 2M$*, which is a contradiction. Therefore, the system* (11)*, when initiated from the set* $\{\, \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_s \mid v(0,\boldsymbol{x}) > \frac{2M}{T} \,\}$*, will reach the set $\mathcal{X}_s$ within the time interval $[0,T]$ and finally the conclusion holds.*

Similar to Lemma 4, a barrier function that relaxes the submartingale requirement (i.e., $\mathcal{L}v(t,\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_s, \forall t \in [0,T]$) in Lemma 5 is formulated in Lemma 6.

**Lemma 6.** *Suppose there exist a barrier function $v(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ and a function $w(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ with $\sup_{(t,\boldsymbol{x})\in[0,T]\times\overline{\mathcal{X}}} |w(t,\boldsymbol{x})| \leq M$ that are continuously differentiable over $t$ and twice continuously differentiable over $\boldsymbol{x}$, satisfying*

$$\begin{cases} \mathcal{L}v(t,\boldsymbol{x}) \geq \alpha v(t,\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \mathcal{X}\setminus\mathcal{X}_s, \forall t \in [0,T] \\ \frac{\partial v(t,\boldsymbol{x})}{\partial t} \geq \alpha v(t,\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \partial\mathcal{X}\cup\partial\mathcal{X}_s, \forall t\in[0,T] \\ v(t,\boldsymbol{x}) \leq 1 + \frac{\partial w(t,\boldsymbol{x})}{\partial t} & \forall \boldsymbol{x} \in \partial\mathcal{X}_s, \forall t \in [0,T] \\ v(t,\boldsymbol{x}) \leq \mathcal{L}w(t,\boldsymbol{x}) & \forall \boldsymbol{x} \in \mathcal{X}\setminus\mathcal{X}_s, \forall t\in[0,T] \\ v(t,\boldsymbol{x}) \leq \frac{\partial w(t,\boldsymbol{x})}{\partial t} & \forall \boldsymbol{x}\in\partial\mathcal{X}, \forall t \in [0,T] \end{cases}, \tag{13}$$

*then,*

$$\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]} \geq \begin{cases} \frac{(\frac{1}{\alpha}v(0,\boldsymbol{x}_0)+\frac{\beta}{\alpha^2})(e^{\alpha T}-1)-\frac{\beta}{\alpha}T}{T} - \frac{2M}{T} & \text{if } \alpha \neq 0 \\ v(0,\boldsymbol{x}_0) + \frac{1}{2}\beta T - \frac{2M}{T} & \text{if } \alpha = 0 \end{cases}$$

*for $\boldsymbol{x}_0 \in \mathcal{X}\setminus\mathcal{X}_s$.*

*Proof.* From (13), we have

$$\begin{cases} \widehat{\mathcal{L}}v(t,\boldsymbol{x}) \geq \alpha v(t,\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \overline{\mathcal{X}\setminus\mathcal{X}_s}, \forall t \in [0,T] \\ v(t,\boldsymbol{x}) \leq 1_{\partial\mathcal{X}_s}(\boldsymbol{x}) + \widehat{\mathcal{L}}w(t,\boldsymbol{x}) & \forall t\in[0,T], \forall \boldsymbol{x} \in \overline{\mathcal{X}\setminus\mathcal{X}_s} \end{cases}.$$

When $\alpha \neq 0$, according to $\widehat{\mathcal{L}}v(t,\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \overline{\mathcal{X}\setminus\mathcal{X}_s}, \forall t \in [0,T]$, we have, for $t\in[0,T]$, that

$$\mathbb{E}[v(t,\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t))] \geq e^{\alpha t}v(0,\boldsymbol{x}_0) + \frac{\beta}{\alpha}(e^{\alpha t}-1).$$

Further, from $v(t,\boldsymbol{x}) \leq 1_{\partial\mathcal{X}_s}(\boldsymbol{x}) + \widehat{\mathcal{L}}w(t,\boldsymbol{x}), \forall t \in [0,T], \forall \boldsymbol{x} \in \overline{\mathcal{X}\setminus\mathcal{X}_s}$, we have

$$\begin{aligned} &\mathbb{P}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T) \in \partial\mathcal{X}_s) \\ &= \mathbb{E}[1_{\partial\mathcal{X}_s}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))] \\ &\geq \frac{\int_0^T \mathbb{E}[1_{\partial\mathcal{X}_s}(\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t))dt]}{T} \text{(according to Lemma 2)} \\ &\geq \frac{\int_0^T \mathbb{E}[v(t,\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t))]dt}{T} - \frac{\mathbb{E}[w(T,\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))] - w(0,\boldsymbol{x}_0)}{T} \\ &\geq \frac{\int_0^T e^{\alpha t}v(0,\boldsymbol{x}_0) + \frac{\beta}{\alpha}(e^{\alpha t}-1)dt}{T} \\ &\qquad - \frac{\mathbb{E}[w(T,\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))] - w(0,\boldsymbol{x}_0)}{T} \\ &\geq \frac{(\frac{1}{\alpha}e^{\alpha t}v(0,\boldsymbol{x}_0) + \frac{\beta}{\alpha^2}e^{\alpha t} - \frac{\beta}{\alpha}t)\mid_0^T}{T} - \frac{2M}{T} \\ &= \frac{(\frac{1}{\alpha}v(0,\boldsymbol{x}_0)+\frac{\beta}{\alpha^2})(e^{\alpha T}-1) - \frac{\beta}{\alpha}T}{T} - \frac{2M}{T}. \end{aligned}$$

The conclusion for $\alpha = 0$ can be obtained via following the above procedure. The proof is completed. ■ □

**Remark 3.** *Similarly, when $\boldsymbol{\sigma}(\cdot) \equiv \boldsymbol{0}$, stochastic system* (1) *degenerates into the deterministic system* (11)*. If there exist a barrier function $v(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n\to\mathbb{R}$ and a function $w(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n \to\mathbb{R}$ with $\sup_{(t,\boldsymbol{x})\in[0,T]\times\overline{\mathcal{X}}}|w(t,\boldsymbol{x})| \leq M$ that are continuously differentiable over $t$ and $\boldsymbol{x}$, satisfying condition* (13)*, then the system* (11)*, when initiated from the set* $\{\, \boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s \mid \frac{(\frac{1}{\alpha}v(0,\boldsymbol{x})+\frac{\beta}{\alpha^2})(e^{\alpha T}-1)-\frac{\beta}{\alpha}T}{T} - \frac{2M}{T} > 0 \,\}$ *(if $\alpha \neq 0$) or* $\{\, \boldsymbol{x} \in \mathcal{X}\setminus\mathcal{X}_s \mid v(0,\boldsymbol{x}) + \frac{1}{2}\beta T - \frac{2M}{T} > 0 \,\}$ *(if $\alpha = 0$), will reach the set $\mathcal{X}_s$ within the time interval $[0,T]$, while remaining within the set $\mathcal{X}$ before the first encounter with $\mathcal{X}_s$.*

Further, a straightforward result can be obtained from Lemma 6 when searching for a time-independent function $v(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$ instead of a time-dependent function $v(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n\to\mathbb{R}$.

**Corollary 2.** *Suppose there exist twice continuously differentiable functions $v(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$ and $w(\boldsymbol{x})\colon\mathbb{R}^n\to\mathbb{R}$ with $\sup_{\boldsymbol{x}\in\overline{\mathcal{X}}}|w(\boldsymbol{x})| \leq M$, satisfying*

$$\begin{cases} \mathcal{L}v(\boldsymbol{x}) \geq \alpha v(\boldsymbol{x}) + \beta & \forall\boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s \\ 0 \geq \alpha v(\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \partial\mathcal{X}\cup\partial\mathcal{X}_s \\ v(\boldsymbol{x}) \leq 1 & \forall\boldsymbol{x}\in\partial\mathcal{X}_s \\ v(\boldsymbol{x}) \leq \mathcal{L}w(\boldsymbol{x}) & \forall \boldsymbol{x}\in\mathcal{X}\setminus\mathcal{X}_s \\ v(\boldsymbol{x}) \leq 0 & \forall\boldsymbol{x}\in\partial\mathcal{X} \end{cases}, \tag{14}$$

*then,*

$$\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]} \geq \begin{cases} \frac{(\frac{1}{\alpha}v(\boldsymbol{x}_0)+\frac{\beta}{\alpha^2})(e^{\alpha T}-1)-\frac{\beta}{\alpha}T}{T} - \frac{2M}{T} & \text{if } \alpha\neq 0 \\ v(\boldsymbol{x}_0) + \frac{1}{2}\beta T - \frac{2M}{T} & \text{if } \alpha = 0 \end{cases}$$

*for $\boldsymbol{x}_0 \in \mathcal{X}\setminus\mathcal{X}_s$.*

**Remark 4.** *When the subset $\mathcal{X}_s$ is equal to the complement of the state constrained set $\mathcal{X}$, i.e., $\mathcal{X}_s = \mathbb{R}^n\setminus\mathcal{X}$, and $\mathcal{X} = \{\, \boldsymbol{x} \mid v(\boldsymbol{x}) < 1 \,\}$ with $\partial\mathcal{X} = \{\, \boldsymbol{x} \mid v(\boldsymbol{x}) = 1\,\}$, a condition* *that lower-bounds the reachability probability is formulated in Theorem 2 in [20], which is presented below,*

$$\begin{cases}\mathcal{L}v(\boldsymbol{x}) \geq \alpha v(\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \mathcal{X} \\ \alpha + \beta > 0 \end{cases}. \tag{15}$$

*Besides, when the $\mathcal{X}_s$ is a subset of the set $\mathcal{X} = \{\, \boldsymbol{x} \mid v(\boldsymbol{x}) > 0 \,\}$ with $\partial\mathcal{X} = \{\, \boldsymbol{x} \mid v(\boldsymbol{x}) = 0 \,\}$ and is equal to $\{\, \boldsymbol{x} \mid v(\boldsymbol{x}) \geq 1 \,\}$, a condition that lower-bounds the reachability probability is formulated in Theorem 1 in [20], which is formulated below,*

$$\begin{cases}\mathcal{L}v(\boldsymbol{x}) \geq \alpha v(\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_s \\ \alpha > -\beta \geq 0 \end{cases}. \tag{16}$$

*When applied to the aforementioned cases in [20], the condition* (14) *complements conditions* (15) *and* (16) *with* $\alpha + \beta \leq 0$.

Besides, we can also obtain similar conclusions as in Remark 2 and 3.

## IV. Bounding Reachability Probabilities II

This section introduces our barrier functions for upper- and lower-bounding the reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^T$ in Definition 3. They are respectively formulated in Subsection IV-A and IV-B.

Since $A \neq B$, where $A = \{\, \boldsymbol{w} \in \Omega \mid \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T) \in \mathcal{X}_s \wedge \forall \tau \in [0,T).\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\tau) \in \mathcal{X} \,\}$ and $B = \{\, \boldsymbol{w} \in \Omega \mid \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T) \in \partial\mathcal{X}_s \wedge \forall \tau \in [0,T).\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\tau) \in \mathcal{X} \,\}$, the construction of the barrier functions in this section lies on a different auxiliary stochastic process $\{\widetilde{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t), t \in \mathbb{R}_{\geq 0}\}$ for $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$ that is a stopped process corresponding to $\{\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t), t \in [0, T^{\boldsymbol{x}_0}(\boldsymbol{w}))\}$ and the set $\overline{\mathcal{X}}$ rather than $\overline{\mathcal{X} \setminus \mathcal{X}_s}$ as $\{\widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t), t \in [0, T^{\boldsymbol{x}_0}(\boldsymbol{w}))\}$, i.e.,

$$\widetilde{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t) = \begin{cases}\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t) & \text{if } t < \tau^{\boldsymbol{x}_0}(\boldsymbol{w}) \\ \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\tau^{\boldsymbol{x}_0}(\boldsymbol{w})) & \text{if } t \geq \tau^{\boldsymbol{x}_0}(\boldsymbol{w}) \end{cases}, \tag{17}$$

where $\tau^{\boldsymbol{x}_0}(\boldsymbol{w}) = \inf\{\, t \mid \boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t) \in \partial\mathcal{X} \,\}$ is the first time of exit of $\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t)$ from the set $\mathcal{X}$. Similar to the stopped process in Section III, the infinitesimal generator corresponding to $\widetilde{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t)$ is identical to the one corresponding to $\boldsymbol{X}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t)$ over $\mathcal{X}$, and is equal to $\frac{\partial v(t,\boldsymbol{x})}{\partial t}$ on $\partial\mathcal{X}$ [9]. That is, for $v(t,\boldsymbol{x})$ which is twice continuously differentiable over $\boldsymbol{x}$ and continuously differentiable over $t$,

$$\begin{aligned}\widetilde{\mathcal{L}}v(t,\boldsymbol{x}) = \mathcal{L}v(t,\boldsymbol{x}) =& \frac{\partial v(t,\boldsymbol{x})}{\partial t} + \frac{\partial v(t,\boldsymbol{x})}{\partial \boldsymbol{x}}\boldsymbol{b}(\boldsymbol{x}) \\ &+ \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}(\boldsymbol{x})^\top \frac{\partial^2 v(t,\boldsymbol{x})}{\partial \boldsymbol{x}^2}\boldsymbol{\sigma}(\boldsymbol{x}))\end{aligned}$$

for $(\boldsymbol{x}, t) \in \mathcal{X} \times [0,T]$ and $\widetilde{\mathcal{L}}v(t,\boldsymbol{x}) = \frac{\partial v(t,\boldsymbol{x})}{\partial t}$ for $\boldsymbol{x} \in \partial\mathcal{X}$ and $t \in [0,T]$.

The exact reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^T$ is equal to the probability of reaching the set $\mathcal{X}_s$ at the time instant $T$ for the above auxiliary stochastic process.

**Lemma 7.** *Given $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$,*

$$\mathbb{P}_{\boldsymbol{x}_0}^T = \mathbb{P}(\widetilde{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T) \in \mathcal{X}_s) = \mathbb{E}[1_{\mathcal{X}_s}(\widetilde{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))].$$

The reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^T$ can also be reduced to a solution to a second-order partial differential equation.

**Theorem 2.** *Suppose there exists a function $v(t,\boldsymbol{x})\colon [0,T] \times \mathbb{R}^n$ which is twice continuously differentiable over $\boldsymbol{x}$ and continuously differentiable over $t$, satisfying*

$$\begin{cases}\widetilde{\mathcal{L}}v(t,\boldsymbol{x}) = 0 & \forall \boldsymbol{x} \in \overline{\mathcal{X}}, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) = 1_{\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X}} \end{cases}, \tag{18}$$

*then, $\mathbb{P}_{\boldsymbol{x}_0}^T = \mathbb{P}(\widetilde{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T) \in \mathcal{X}_s) = v(0,\boldsymbol{x}_0)$ for $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$.*

*Proof.* The proof is similar to Theorem 1. ■ □

### A. Upper-bounding Reachability Probabilities

In this subsection, we present our barrier functions for upper-bounding the reachability probability, denoted as $\mathbb{P}_{\boldsymbol{x}_0}^T$ in Definition 3. The first time-dependent barrier function is obtained via relaxing equation (18) in Theorem 2. The second one extends upon the first one, which relaxes the supermartingale requirement. The third one is a variant of the second one, using a time-independent function $v(\boldsymbol{x})$ instead of a time-dependent function $v(t,\boldsymbol{x})$. They are respectively formulated in Lemma 8, 9, and Corollary 3.

**Lemma 8.** *Suppose there exists a barrier function $v(t,\boldsymbol{x})\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable over $t$ and twice continuously differentiable over $\boldsymbol{x}$, satisfying*

$$\begin{cases}\mathcal{L}v(t,\boldsymbol{x}) \leq 0 & \forall \boldsymbol{x} \in \mathcal{X}, \forall t \in [0,T] \\ \frac{\partial v(t,\boldsymbol{x})}{\partial t} \leq 0 & \forall \boldsymbol{x} \in \partial\mathcal{X}, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) \geq 1_{\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X}} \end{cases}, \tag{19}$$

*then, $\mathbb{P}_{\boldsymbol{x}_0}^T \leq v(0,\boldsymbol{x}_0)$ for $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$.*

*Proof.* From (19), we have

$$\begin{cases}\widetilde{\mathcal{L}}v(t,\boldsymbol{x}) \leq 0 & \forall \boldsymbol{x} \in \overline{\mathcal{X}}, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) \geq 1_{\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X}} \end{cases}. \tag{20}$$

Then, we can obtain the conclusion via using Lemma 7 and following the proof of Lemma 3. ■ □

Lemma 8 states that if there exists a barrier function $v(t,\boldsymbol{x})\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfying (19), then the reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^T$ can be bounded above by $v(0,\boldsymbol{x}_0)$. Analogously, the requirement for $\mathcal{L}v(t,\boldsymbol{x}) \leq 0$ to hold for $(t,\boldsymbol{x}) \in [0,T] \times \mathcal{X}$ may hinder the acquisition of such a barrier function. We will relax this requirement below.

**Lemma 9.** *Suppose there exists a barrier function $v(t,\boldsymbol{x})\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable over $t$ and twice continuously differentiable over $\boldsymbol{x}$, satisfying*

$$\begin{cases}\mathcal{L}v(t,\boldsymbol{x}) \leq \alpha v(t,\boldsymbol{x}) + \beta, & \forall \boldsymbol{x} \in \mathcal{X}, \forall t \in [0,T] \\ \frac{\partial v(t,\boldsymbol{x})}{\partial t} \leq \alpha v(t,\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \partial\mathcal{X}, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) \geq 1_{\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X}} \end{cases}, \tag{21}$$

*then,*

$$\mathbb{P}_{\boldsymbol{x}_0}^T \leq \begin{cases} v(0,\boldsymbol{x}_0) + \beta T & \textit{if } \alpha = 0 \\ e^{\alpha T} v(0,\boldsymbol{x}_0) + \frac{\beta}{\alpha}(e^{\alpha T} - 1) & \textit{if } \alpha \neq 0 \end{cases}$$

*for $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$.*

*Proof.* From (21), we have

$$\begin{cases} \widetilde{\mathcal{L}}v(t,\boldsymbol{x}) \le \alpha v(t,\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \overline{\mathcal{X}}, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) \ge 1_{\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X}} \end{cases}.$$

Then, we can obtain the conclusion via using Lemma 7 and following the proof of Lemma 4. ■ □

A straightforward result is obtained from Lemma 9 when searching for a time-independent barrier function $v(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$ instead of a time-dependent one $v(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$.

**Corollary 3.** *Suppose there exists a barrier function $v(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$, which is twice continuously differentiable, satisfying*

$$\begin{cases} \mathcal{L}v(\boldsymbol{x}) \le \alpha v(\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \mathcal{X} \\ 0 \le \alpha v(\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \partial\mathcal{X} \\ v(\boldsymbol{x}) \ge 1_{\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X}} \end{cases}, \quad (22)$$

*then,*

$$\mathbb{P}_{\boldsymbol{x}_0}^T \le \begin{cases} v(\boldsymbol{x}_0) + \beta T & \text{if } \alpha = 0 \\ e^{\alpha T} v(\boldsymbol{x}_0) + \frac{\beta}{\alpha}(e^{\alpha T} - 1) & \text{if } \alpha \ne 0 \end{cases}$$

*for $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$.*

### B. Lower-bounding Reachability Probabilities

In this subsection, we present our barrier functions for lower-bounding the reachability probability $\mathbb{P}_{\boldsymbol{x}_0}^T$, based on the following assumption on $\mathcal{X}_s$.

**Assumption 2.** *The set $\mathcal{X}_s$ has non-empty interior, i.e., $\mathcal{X}_s^\circ \ne \emptyset$.*

Like the one in Lemma 8, the construction of the first barrier function was obtained by relaxing equation (18).

**Lemma 10.** *Suppose there exists a barrier function $v(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable over $t$ and twice continuously differentiable over $\boldsymbol{x}$, satisfying*

$$\begin{cases} \mathcal{L}v(t,\boldsymbol{x}) \ge 0 & \forall \boldsymbol{x} \in \mathcal{X}, \forall t \in [0,T] \\ \frac{\partial v(t,\boldsymbol{x})}{\partial t} \ge 0 & \forall \boldsymbol{x} \in \partial\mathcal{X}, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) \le 1_{\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X}} \end{cases}, \quad (23)$$

*then, $\mathbb{P}_{\boldsymbol{x}_0}^T \ge v(0,\boldsymbol{x}_0)$ for $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$.*

*Proof.* From (23), we have

$$\begin{cases} \widetilde{\mathcal{L}}v(t,\boldsymbol{x}) \ge 0 & \forall \boldsymbol{x} \in \overline{\mathcal{X}}, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) \le 1_{\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X}} \end{cases}.$$

Then, we can obtain the conclusion via using Lemma 7 and following the proof of Lemma 3. ■ □

**Remark 5.** *The barrier function for certifying reach-avoid analysis at the time instant $T$ (i.e., reach the set $\mathcal{X}_r$ at the time instant $t = T$ while staying within the set $\mathcal{X}$ before $T$) for deterministic systems can be retrieved from the one satisfying* (23)*.*

*When $\boldsymbol{\sigma}(\cdot) \equiv 0$, if there exists a barrier function $v(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable over $t$ and $\boldsymbol{x}$, satisfying condition* (23)*, then the system described by equation* (11)*, when initiated from the set $\{\, \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_s \mid v(0,\boldsymbol{x}) > 0 \,\}$, will reach the set $\mathcal{X}_s$ at $t = T$, while remaining within the set $\mathcal{X}$ before $t = T$. This problem has been studied in [21]. A proof of this conclusion is shown here: we first demonstrate that if system* (11) *is initiated from the set $\{\, \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_s \mid v(0,\boldsymbol{x}) > 0 \,\}$, it will stay within the set $\mathcal{X}$ over the time horizon $[0,T]$. Assume this is not true. Then there exists $\tau \in [0,T]$ such that $\boldsymbol{X}_{\boldsymbol{x}_0}(\tau) \in \partial\mathcal{X}$ and $\boldsymbol{X}_{\boldsymbol{x}_0}(t) \in \mathcal{X}$ for $t \in [0,\tau)$. Since $\mathcal{L}v(t,\boldsymbol{x}) \ge 0$ for $t \in [0,T]$ and $\boldsymbol{x} \in \mathcal{X}$, we have*

$$v(\tau, \boldsymbol{X}_{\boldsymbol{x}_0}(\tau)) \ge v(0,\boldsymbol{x}_0) > 0.$$

*Further, since $\frac{\partial v(t,\boldsymbol{x})}{\partial t} \ge 0$ for $(t,\boldsymbol{x}) \in [0,T]\times\partial\mathcal{X}$, we have*

$$v(T, \boldsymbol{X}_{\boldsymbol{x}_0}(\tau)) \ge v(\tau, \boldsymbol{X}_{\boldsymbol{x}_0}(\tau)) > 0,$$

*which contradicts $v(T,\boldsymbol{x}) \le 1_{\mathcal{X}_s}(\boldsymbol{x}), \forall \boldsymbol{x} \in \overline{\mathcal{X}}$. Therefore, the system* (11)*, when initiated from the set $\{\, \boldsymbol{x} \in \mathcal{X}_s \mid v(0,\boldsymbol{x}) > 0 \,\}$, stays within the set $\mathcal{X}$ over the time horizon $[0,T]$. Next, we show that the system* (11)*, when initiated from the set $\{\, \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_s \mid v(0,\boldsymbol{x}) > 0 \,\}$, will reach the set $\mathcal{X}_s$ at $t = T$. Assume that this conclusion does not hold, which indicates that the system, when initiated from the set $\{\, \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_s \mid v(0,\boldsymbol{x}) > 0 \,\}$, will stay within the set $\mathcal{X} \setminus \mathcal{X}_s$ over the time horizon $[0,T]$. From $\mathcal{L}v(t,\boldsymbol{x}) \ge 0$ for $(t,\boldsymbol{x}) \in [0,T]\times\mathcal{X}\setminus\mathcal{X}_s$, we can obtain $v(T,\boldsymbol{X}_{\boldsymbol{x}_0}(T)) > 0$, which contradicts $v(T,\boldsymbol{x}) \le 1_{\mathcal{X}_s}(\boldsymbol{x}), \forall \boldsymbol{x} \in \overline{\mathcal{X}}$. Therefore, the system* (11)*, when initiated from the set $\{\, \boldsymbol{x} \in \mathcal{X}_s \mid v(0,\boldsymbol{x}) > 0 \,\}$, will reach the set $\mathcal{X}_s$ at $t = T$ and finally the conclusion holds.*

Based on the stochastic process $\widetilde{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(\cdot)$, one might wander the possibility of constructing a barrier function, similar to the one in Lemma 10, to lower-bound $\mathbb{P}_{\boldsymbol{x}_0}^T$. Let us consider the existence of a barrier function $v(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ that is twice continuously differentiable with respect to $\boldsymbol{x}$ and continuously differentiable over $t$, satisfying

$$\begin{cases} \mathcal{L}v(t,\boldsymbol{x}) \ge 0 & \forall \boldsymbol{x} \in \mathcal{X}, \forall t \in [0,T] \\ \frac{\partial v(t,\boldsymbol{x})}{\partial t} \ge 0 & \forall \boldsymbol{x} \in \mathcal{X}_s, \forall t \in [0,T] \\ \frac{\partial v(t,\boldsymbol{x})}{\partial t} \ge 0 & \forall \boldsymbol{x} \in \partial\mathcal{X}, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) \le 1_{\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X}} \end{cases}. \quad (24)$$

We can conclude that $\mathbb{P}_{\boldsymbol{x}_0}^{[0,T]} \ge v(0,\boldsymbol{x}_0)$ for $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$. However, it is observed that if $v(\boldsymbol{x})$ satisfies (24), it also satisfies (10) with $w(\boldsymbol{x}) \equiv 0$. As commented in Remark 1, we cannot obtain meaningless results.

Similar to Lemma 9, a condition that relaxes the submartingale requirement (i.e., $\mathcal{L}v(t,\boldsymbol{x}) \ge 0, \forall \boldsymbol{x} \in \mathcal{X}, \forall t \in [0,T]$) in Lemma 10 is formulated in Lemma 11.

**Lemma 11.** *Suppose there exist a barrier function $v(t,\boldsymbol{x})\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable over $t$ and twice continuously differentiable over $\boldsymbol{x}$, satisfying*

$$\begin{cases} \mathcal{L}v(t,\boldsymbol{x}) \ge \alpha v(t,\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \mathcal{X}, \forall t \in [0,T] \\ \frac{\partial v(t,\boldsymbol{x})}{\partial t} \ge \alpha v(t,\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \partial\mathcal{X}, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) \le 1_{\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X}} \end{cases}, \quad (25)$$

*then,*

$$\mathbb{P}_{\boldsymbol{x}_0}^T \ge \begin{cases} e^{\alpha T} v(0,\boldsymbol{x}_0) + \frac{\beta}{\alpha}(e^{\alpha T} - 1) & \text{if } \alpha \ne 0 \\ v(0,\boldsymbol{x}_0) + \beta T & \text{if } \alpha = 0 \end{cases}$$

*for* $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$.

*Proof.* From (25), we have

$$\begin{cases} \widetilde{\mathcal{L}}v(t,\boldsymbol{x}) \geq \alpha v(t,\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \overline{\mathcal{X}}, \forall t \in [0,T] \\ v(T,\boldsymbol{x}) \leq 1_{\partial \mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X}} \end{cases}.$$

If $\alpha \neq 0$, according to $\widetilde{\mathcal{L}}v(t,\boldsymbol{x}) \geq \alpha v(t,\boldsymbol{x}) + \beta, \forall \boldsymbol{x} \in \overline{\mathcal{X}}, \forall t \in [0,T]$, we have, for $t \in [0,T]$, that

$$\mathbb{E}[v(t, \widetilde{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(t))] \geq e^{\alpha t} v(0,\boldsymbol{x}_0) + \frac{\beta}{\alpha}(e^{\alpha t} - 1).$$

Further, from $v(T,\boldsymbol{x}) \leq 1_{\partial \mathcal{X}_s}(\boldsymbol{x}), \forall \boldsymbol{x} \in \overline{\mathcal{X}}$, we have

$$\begin{aligned} \mathbb{P}(\widetilde{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T) \in \mathcal{X}_s) &= \mathbb{E}[1_{\mathcal{X}_s}(\widetilde{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))] \\ &\geq \mathbb{E}[v(T, \widehat{\boldsymbol{X}}_{\boldsymbol{x}_0}^{\boldsymbol{w}}(T))] \\ &\geq e^{\alpha T} v(0,\boldsymbol{x}_0) + \frac{\beta}{\alpha}(e^{\alpha T} - 1). \end{aligned}$$

If $\alpha = 0$, we can obtain $\mathbb{P}_{\boldsymbol{x}_0}^T \geq v(0,\boldsymbol{x}_0) + \beta T$ by following the above procedure.

The proof is completed. ■ □

**Remark 6.** *Similarly, when* $\boldsymbol{\sigma}(\cdot) \equiv \mathbf{0}$*, stochastic system* (1) *degenerates into the deterministic system* (11)*. If there exists a barrier function* $v(t,\boldsymbol{x})\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *that is continuously differentiable over* $t$ *and* $\boldsymbol{x}$*, satisfying condition* (25)*, then the system* (11)*, when initiated from the set* $\{\, \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_s \mid e^{\alpha T} v(0,\boldsymbol{x}) + \frac{\beta}{\alpha}(e^{\alpha T} - 1) > 0 \,\}$ *(if* $\alpha \neq 0$*) or* $\{\, \boldsymbol{x} \in \mathcal{X} \setminus \mathcal{X}_s \mid v(0,\boldsymbol{x}) + \beta T > 0 \,\}$ *(if* $\alpha = 0$*), will reach the set* $\mathcal{X}_s$ *at* $t = T$*, while remaining within the set* $\mathcal{X}$ *before the time instant* $T$*.*

Further, a straightforward result can be obtained from Lemma 11 when searching for a time-independent function $v(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$ instead of a time-dependent function $v(t,\boldsymbol{x})\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$.

**Corollary 4.** *Suppose there exists a twice continuously differentiable function* $v(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$*, satisfying*

$$\begin{cases} \mathcal{L}v(\boldsymbol{x}) \geq \alpha v(\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \mathcal{X} \\ 0 \geq \alpha v(\boldsymbol{x}) + \beta & \forall \boldsymbol{x} \in \partial \mathcal{X} \\ v(\boldsymbol{x}) \leq 1_{\mathcal{X}_s}(\boldsymbol{x}) & \forall \boldsymbol{x} \in \overline{\mathcal{X}} \end{cases}, \tag{26}$$

*then,*

$$\mathbb{P}_{\boldsymbol{x}_0}^T \geq \begin{cases} e^{\alpha T} v(\boldsymbol{x}_0) + \frac{\beta}{\alpha}(e^{\alpha T} - 1) & \text{if } \alpha \neq 0 \\ v(\boldsymbol{x}_0) + \beta T & \text{if } \alpha = 0 \end{cases}$$

*for* $\boldsymbol{x}_0 \in \mathcal{X} \setminus \mathcal{X}_s$.

Besides, we can obtain similar conclusion as in Remark 6.

## V. Conclusion

This paper introduced a new framework of constructing barrier functions to establish both upper and lower bounds on the probabilities of reaching specific sets over finite time horizons and at finite time instants in continuous-time stochastic systems described by SDEs. These proposed barrier functions offer stronger alternatives, complement existing methods, or fill gaps, facilitating the calculation of precise bounds on reachability probabilities.

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